Linear Algebra II

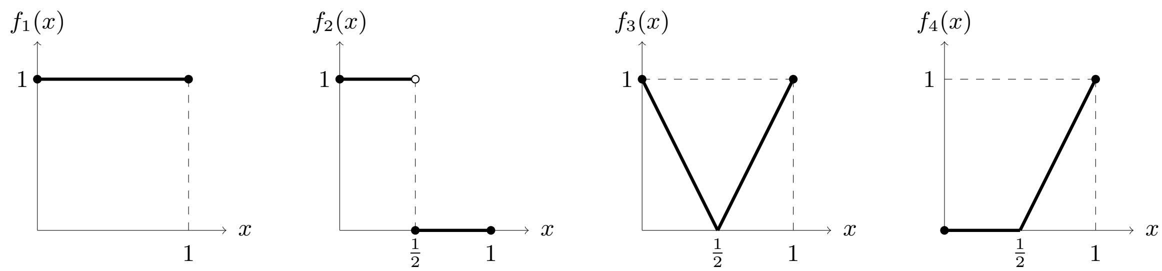
19/03/2015, Thursday, 14:00-16:00

1 $(3 + 8 + 3 + 8 + 8 = 30 \text{ pts})$ **Inner product spaces**

Consider the vector space $C[0,1]$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Let f_1, f_2, f_3 and f_4 be given by as follows:



(a) Is the set $\{f_1, f_2, f_3\}$ an orthonormal set?

(b) Compute $\langle f_i, f_j\rangle$ where $i, j \in \{1, 2, 3, 4\}$. (Hint: You may want to use the relationship between the integral of and the area under a curve.)

(c) Find the angle between f_1 and f_2.

(d) Apply Gram-Schmidt process to obtain an orthonormal basis for the subspace spanned by f_1, f_2 and f_3. (Hint: You may want to plot each function the process computes.)

(e) Find the closest function to f_4 in the subspace spanned by f_1, f_2 and f_3.

Required Knowledge: **Orthogonality, Gram-Schmidt process, least-squares approximation.**

Solution:

1a: Note that

$$\langle f_1, f_2\rangle = \int_0^1 f_1(x)f_2(x)\,dx = \frac{1}{2}.$$

As such, this set is *not* an orthonormal set.

1b: Straightforward calculations yield:

$$\begin{aligned}
\langle f_1, f_1 \rangle &= \int_0^1 dx = 1 \\
\langle f_1, f_2 \rangle &= \int_0^{\frac{1}{2}} dx = \frac{1}{2} \\
\langle f_1, f_3 \rangle &= \int_0^1 f_3(x)\, dx = \frac{1}{2} \\
\langle f_1, f_4 \rangle &= \int_0^1 f_4(x)\, dx = \frac{1}{4} \\
\langle f_2, f_2 \rangle &= \int_0^{\frac{1}{2}} dx = \frac{1}{2} \\
\langle f_2, f_3 \rangle &= \int_0^{\frac{1}{2}} f_3(x)\, dx = \frac{1}{4} \\
\langle f_2, f_4 \rangle &= \int_0^{\frac{1}{2}} f_4(x)\, dx = 0.
\end{aligned}$$

Note that

$$f_3(x) = \begin{cases} -2x+1 & \text{if } 0 \leqslant x \leqslant \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

This results in

$$\begin{aligned}
\langle f_3, f_3 \rangle &= \int_0^{\frac{1}{2}} (-2x+1)^2\, dx + \int_{\frac{1}{2}}^1 (2x-1)^2\, dx \\
&= \int_0^1 (2x-1)^2\, dx \\
&= (4\frac{x^3}{3} - 2x^2 + x)\Big|_0^1 \\
&= \frac{4}{3} - 2 + 1 = \frac{1}{3} \\
\langle f_3, f_4 \rangle &= \int_{\frac{1}{2}}^1 (2x-1)^2\, dx \\
&= \frac{1}{6} \\
\langle f_4, f_4 \rangle &= \int_{\frac{1}{2}}^1 (2x-1)^2 \\
&= \frac{1}{6}.
\end{aligned}$$

1c: The angle θ between f_1 and f_2 is defined by

$$\cos(\theta) = \frac{\langle f_1, f_2 \rangle}{\|f_1\| \|f_2\|}.$$

Thus, we get

$$\cos\theta = \frac{\frac{1}{2}}{1 \cdot \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{2}.$$

As such, the angle between these two functions is 45 degrees.

1d: By applying the Gram-Schmidt process, we obtain:

$$u_1 = \frac{f_1}{\|f_1\|}$$

$$u_1 = f_1$$

$$u_2 = \frac{f_2 - p_1}{\|f_2 - p_1\|}$$

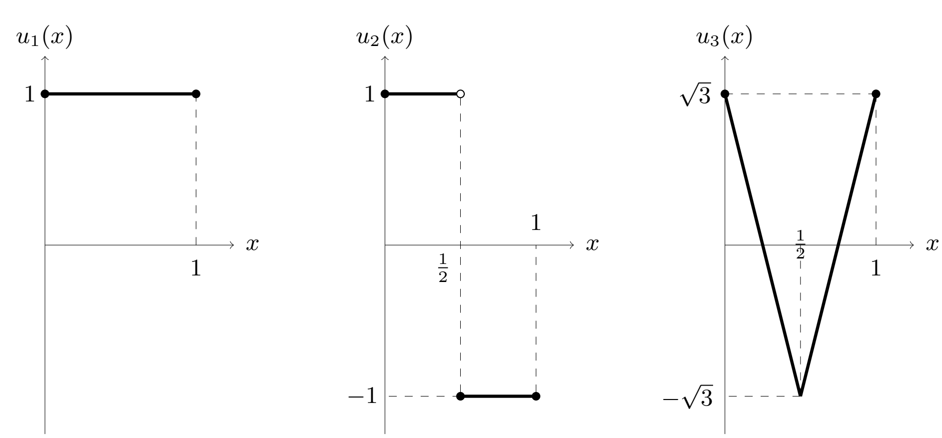
$$\begin{aligned} p_1 &= \langle f_2, f_1\rangle \cdot f_1 \\ &= \frac{1}{2} f_1 \\ f_2 - p_1 &= f_2 - \frac{1}{2} f_1 \\ \|f_2 - \frac{1}{2}f_1\|^2 &= \langle f_2 - \frac{1}{2}f_1, f_2 - \frac{1}{2}f_1\rangle = \langle f_2, f_2\rangle - 2\cdot\frac{1}{2}\langle f_2, f_1\rangle + \frac{1}{4}\langle f_1, f_1\rangle \\ \|f_2 - \frac{1}{2}f_1\|^2 &= \frac{1}{2} - \frac{1}{2} + \frac{1}{4} = \frac{1}{4} \\ \|f_2 - \frac{1}{2}f_1\| &= \frac{1}{2} \end{aligned}$$

$$u_2 = 2(f_2 - \frac{1}{2}f_1) = 2f_2 - f_1$$

$$u_3 = \frac{f_3 - p_2}{\|f_3 - p_2\|}$$

$$\begin{aligned} p_2 &= \langle f_3, f_1\rangle f_1 + \langle f_3, 2f_2 - f_1\rangle (2f_2 - f_1) \\ &= \frac{1}{2} f_1 + (2\cdot\frac{1}{4} - \frac{1}{2})(2f_2 - f_1) \\ &= \frac{1}{2} f_1 \\ f_3 - p_2 &= f_3 - \frac{1}{2} f_1 \\ \|f_3 - \frac{1}{2}f_1\|^2 &= \langle f_3 - \frac{1}{2}f_1, f_3 - \frac{1}{2}f_1\rangle \\ \|f_3 - \frac{1}{2}f_1\|^2 &= \langle f_3, f_3\rangle - 2\cdot\frac{1}{2}\langle f_3, f_1\rangle + \frac{1}{4}\langle f_1, f_1\rangle \\ \|f_3 - \frac{1}{2}f_1\|^2 &= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12} \\ \|f_3 - \frac{1}{2}f_1\| &= \frac{1}{2\sqrt{3}} \end{aligned}$$

$$u_3 = 2\sqrt{3}(f_3 - \frac{1}{2}f_1).$$



1e: The closest function is given by

$$
\begin{aligned}
p &= \langle f_4, u_1\rangle u_1 + \langle f_4, u_2\rangle u_2 + \langle f_4, u_3\rangle u_3 \\
&= \langle f_4, f_1\rangle f_1 + \langle f_4, 2f_2 - f_1\rangle (2f_2 - f_1) + \langle f_4, 2\sqrt{3}(f_3 - \frac{1}{2}f_1)\rangle 2\sqrt{3}(f_3 - \frac{1}{2}f_1) \\
&= \frac{1}{4}f_1 + (2\cdot 0 - \frac{1}{4})(2f_2 - f_1) + 12(\frac{1}{6} - \frac{1}{8})(f_3 - \frac{1}{2}f_1) \\
&= \frac{1}{4}f_1 - \frac{1}{2}f_2 + \frac{1}{4}f_1 + \frac{1}{2}f_3 - \frac{1}{4}f_1 \\
&= \frac{1}{4}f_1 - \frac{1}{2}f_2 + \frac{1}{2}f_3.
\end{aligned}
$$

(a) Let A be a square matrix and p be a polynomial. Show that if x is an eigenvector of A corresponding the eigenvalue of λ then x is also an eigenvector of $p(A)$. Find the corresponding eigenvalue.

(b) Let A and B be nonsingular matrices of the same size. Show that AB and BA have the same eigenvalues.

REQUIRED KNOWLEDGE: **Eigenvalues, eigenvectors, similarity.**

SOLUTION:

2a: Since x is an eigenvector corresponding to the eigenvalue λ, we have

$$Ax = \lambda x.$$

By pre-multiplying by A, we get

$$A^2 x = A(\lambda x) = \lambda(Ax) = \lambda^2 x.$$

Repeating this argument yields

$$A^k x = \lambda^k x$$

for any positive integer k. Now, let p be given by

$$p(s) = p_\ell s^\ell + p_{\ell-1} s^{\ell-1} + \cdots + p_1 s + p_0.$$

Note that

$$\begin{aligned} p(A)x &= (p_\ell A^\ell + p_{\ell-1} A^{\ell-1} + \cdots + p_1 A + p_0 I)x \\ &= p_\ell A^\ell x + p_{\ell-1} A^{\ell-1} x + \cdots + p_1 Ax + p_0 x \\ &= (p_\ell \lambda^\ell + p_{\ell-1} \lambda^{\ell-1} + \cdots + p_1 \lambda + p_0)x \\ &= p(\lambda)x. \end{aligned}$$

Therefore, x is an eigenvector of $p(A)$ corresponding to the eigenvalue $p(\lambda)$.

2b: Note that

$$BA = B(AB)B^{-1}.$$

As such, AB and BA are similar matrices. Consequently, they share the same eigenvalues.

(a) Let A be a nonsingular matrix. Show that if A is diagonalizable then so is A^{-1}.

(b) Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, and d are real numbers. Determine all values of (a, b, c, d) such that M is unitarily diagonalizable.

(c) Let

$$A = \begin{bmatrix} p & q \\ -q & p \end{bmatrix}$$

where p and q are real numbers. Find a unitary diagonalizer for A.

REQUIRED KNOWLEDGE: **Diagonalization, unitary matrices, normal matrices.**

SOLUTION:

3a:

Approach 1: Observe that if (λ, x) is an eigenpair of the matrix A then (λ^{-1}, x) is an eigenpair of A^{-1}. This means that the number of linearly independent eigenvectors of A is equal to that of A^{-1}. As A is diagonalizable, so must be A^{-1}.

Approach 2: If A is diagonalizable, there exist a nonsingular matrix T and a diagonal matrix D such that

$$A = TDT^{-1}.$$

Since A is nonsingular, D must be nonsingular too. By inverting A, we get

$$A^{-1} = (TDT^{-1})^{-1} = TD^{-1}T^{-1}.$$

Since the inverse of a diagonal matrix is itself a diagonal matrix, we can conclude that A^{-1} diagonalizable.

3b: A matrix M is unitarily diagonalizable if and only if it is normal, that is $M^H M = MM^H$. Note that

$$M^T M = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix} \quad \text{and} \quad MM^T = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}.$$

As such, M is diagonalizable if and only if

$$a^2 + c^2 = a^2 + b^2, \quad ab + cd = ac + bd, \quad \text{and} \quad b^2 + d^2 = c^2 + d^2.$$

These are equivalent to

$$b^2 = c^2 \quad \text{and} \quad ab + cd = ac + bd.$$

The first holds if and only if $b = c$ or $b = -c$. Substituting these into the second, we get the following condition for unitarily diagonalizability of M:

$$(b = c) \quad \text{OR} \quad (b = -c \ \text{ AND } \ a = d).$$

3c: It follows from the last conclusion that A is unitarily diagonalizable. In order to find such a diagonalizer, we distinguish two cases.

$\mathbf{q = 0}$**:** In this case, A is already diagonal. So, the 2×2 identity matrix is a unitary diagonalizer.

q ≠ 0: We begin first calculating the characteristic polynomial:

$$\det(A - \lambda I) = \det(\begin{bmatrix} p-\lambda & q \\ -q & p-\lambda \end{bmatrix}) = (p-\lambda)^2 + q^2.$$

Note that

$$(p-\lambda)^2 + q^2 = 0 \quad \text{if and only if} \quad p - \lambda = \pm iq.$$

Therefore, the eigenvalues are given by $\lambda_1 = p + iq$ and $\lambda_2 = p - iq$. Next, we continue with finding the eigenvectors.

For $\lambda_1 = p + iq$, we need to solve

$$\begin{bmatrix} -iq & q \\ -q & -iq \end{bmatrix} x = 0.$$

Since $q \neq 0$, we have

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} x = 0.$$

This results in

$$x = \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

For $\lambda_1 = p - iq$, we need to solve

$$\begin{bmatrix} iq & q \\ -q & iq \end{bmatrix} y = 0.$$

Since $q \neq 0$, we have

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} y = 0.$$

This results in

$$y = \begin{bmatrix} -1 \\ i \end{bmatrix}.$$

Note that the matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix}$$

is a unitary matrix and also that

$$\begin{bmatrix} p & q \\ -q & p \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \begin{bmatrix} p+iq & 0 \\ 0 & p-iq \end{bmatrix}.$$

(a) Compute a singular value decomposition of the matrix

$$M = \begin{bmatrix} 2 & -4 \\ 2 & 2 \\ -4 & 0 \\ 1 & 4 \end{bmatrix}.$$

(b) Find the closest (with respect to Frobenius norm) matrix of rank 1 to M.

REQUIRED KNOWLEDGE: **Singular value decomposition, lower rank approximation.**

SOLUTION:

4a: Note that

$$M^T M = \begin{bmatrix} 2 & 2 & -4 & 1 \\ -4 & 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 2 & 2 \\ -4 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 36 \end{bmatrix}.$$

Therefore, the eigenvalues of $M^T M$ are given by

$$\lambda_1 = 36 \quad \text{and} \quad \lambda_2 = 25$$

and hence the singular values of M are given by

$$\sigma_1 = 6 \quad \text{and} \quad \sigma_2 = 5.$$

Note that

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are the normalised eigenvectors for λ_1 and λ_2, respectively. This results in

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let

$$u_1 = \frac{1}{\sigma_1} M v_1 = \frac{1}{6} \begin{bmatrix} -4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

and

$$u_2 = \frac{1}{\sigma_2} M v_2 = \frac{1}{5} \begin{bmatrix} 2 \\ 2 \\ -4 \\ 1 \end{bmatrix}.$$

Now, we need to find an orthonormal basis for $\mathcal{N}(M^T)$. Consider the linear system

$$\begin{bmatrix} 2 & 2 & -4 & 1 \\ -4 & 2 & 0 & 4 \end{bmatrix} w = 0.$$

Clearly, it is equivalent to

$$\begin{bmatrix} 2 & 2 & -4 & 1 \\ 0 & 6 & -8 & 6 \end{bmatrix} w = 0$$

Two independent solutions can be given by

$$w_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad w_2 = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix}.$$

To orthonormalize, we can apply Gram-Schmidt process:

$$u_3 = \frac{w_1}{\|w_1\|}$$

$$u_3 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix}$$

$$u_4 = \frac{w_2 - p_1}{\|w_2 - p_1\|} \qquad p_1 = \langle w_2, u_3 \rangle \cdot u_3$$

$$= -2 \cdot \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 4 \\ 0 \\ -4 \end{bmatrix}$$

$$w_2 - p_1 = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -2 \\ 4 \\ 0 \\ -4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8 \\ 8 \\ -1 \\ 4 \end{bmatrix}$$

$$\|w_2 - \frac{1}{2}p_1\|^2 = \frac{1}{9}(64 + 64 + 1 + 16) = \frac{145}{9}$$

$$u_4 = \frac{1}{\sqrt{145}}\begin{bmatrix} 8 \\ 8 \\ -1 \\ 4 \end{bmatrix}.$$

Thus, we have the SVD:

$$\begin{bmatrix} 2 & -4 \\ 2 & 2 \\ -4 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{2}{5} & \frac{1}{3} & \frac{8}{\sqrt{145}} \\ \frac{1}{3} & \frac{2}{5} & -\frac{2}{3} & \frac{8}{\sqrt{145}} \\ 0 & -\frac{4}{5} & 0 & -\frac{1}{\sqrt{145}} \\ \frac{2}{3} & \frac{1}{5} & \frac{2}{3} & \frac{4}{\sqrt{145}} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

4b: The best rank 1 approximation is given by:

$$X = \begin{bmatrix} -\frac{2}{3} & \frac{2}{5} & \frac{1}{3} & \frac{8}{\sqrt{145}} \\ \frac{1}{3} & \frac{2}{5} & -\frac{2}{3} & \frac{8}{\sqrt{145}} \\ 0 & -\frac{4}{5} & 0 & -\frac{1}{\sqrt{145}} \\ \frac{2}{3} & \frac{1}{5} & \frac{2}{3} & \frac{4}{\sqrt{145}} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 0 & 2 \\ 0 & 0 \\ 0 & 4 \end{bmatrix}.$$
